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# Radiation energy loss of an accelerated charge in an absorbing medium

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**Abstract**

The radiation produced by an arbitrarily moving relativistic charge in an absorbing medium is considered in terms of the radiation energy loss. The energy-angular distributions of emitted photons for Cherenkov radiation, prompt bremsstrahlung, and uniform motion over finite distance (the Tamm problem) are derived. The Doppler effect in an absorbing medium for undulator-like motion is considered. The high-energy, low-angle approximation of the spectra is discussed in terms of the complex formation zone of the radiation produced by an ultra-relativistic charge in an absorbing medium.


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## 1. Introduction. Poynting's theorem in an absorbing medium

Relativistic particle energy loss in a medium can be subdivided into the radiation (RL, $\perp$) and polarisation (PL, $\parallel$) losses, i.e., the work done by the particle against the transverse and longitudinal (relative to the wave vector) components of its electric field at the current particle position, respectively, [1,2]. In terms of a quantum approach, RL and PL are responsible for the generation of transverse (photons in the medium) and longitudinal (plasmons, delta-electrons) medium excitations along the particle trajectory, respectively.

RL, reflecting the contribution from the particle electric current, is responsible for the relativistic dependence of the particle energy loss. It results in observable effects that are used for charged particle identification, i.e., soft (non-destructive for the particle momentum) estimation of the particle Lorentz factor. In the visible and ultraviolet ranges of transferred energies, RL generalises Cherenkov radiation for the case when medium absorption is taken

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into account. In the range of atomic frequencies, where the generated photons are absorbed in the vicinity of the particle trajectory resulting in measured ionisation, RL describes the relativistic rise of $dE/dx$. In the X-ray range, where RL in a uniform medium is suppressed by destructive interference, introduction of proper distributed medium interfaces recovers the generation of X-ray photons. The detection of these photons (X-ray transition radiation) allows one to extend the relativistic dependence of the particle energy loss.

Consideration of the relativistic dependence of the particle energy loss is convenient from the simulation point of view, allowing one to track (involving all background processes) secondary photons from the points of their generation. The GEANT4 object-oriented toolkit for simulation in high-energy physics [3] provides extended tools for simulation of particle identification processes in complex detector geometries. The aim of the present Letter is to discuss the concept of RL and examples of its application for the different laws of charge motion in a form which is convenient for implementations in the framework of GEANT4.

Let us consider a relativistic charged particle with the charge $e$ moving in an absorbing medium with complex dielectric permittivity $\epsilon = \epsilon_1 + i\epsilon_2$. Poynting's theorem can be derived directly from Maxwell's equations [4,5]:

$$-\mathbf{jE} = \frac{1}{4\pi}\left\{\mathbf{E}\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}\frac{\partial \mathbf{B}}{\partial t}\right\} + \operatorname{div}\mathbf{S}, \tag{1}$$

where $\mathbf{E}$ and $\mathbf{D}$ are the electric field and displacement, $\mathbf{H}$ and $\mathbf{B}$ are the magnetic field and induction, respectively, $\mathbf{j} = e\mathbf{v}\delta(\mathbf{r} - \mathbf{v}t)$ is the current density of the external (for simplicity, point-like) charge ($\mathbf{r}$ is the particle coordinate at the time moment $t$, and $\delta$ is the Dirac delta function), and (in the Gauss system of units):

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H},$$

is the Poynting vector expressing the electro-magnetic energy flux density ($c$ is the speed of light in vacuum). Integration of (1) over a large volume $V$ surrounding the particle trajectory results in:

$$-e\mathbf{v}\mathbf{E}(\mathbf{v}t, t) = \frac{1}{4\pi}\int_V \left\{\mathbf{E}\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}\frac{\partial \mathbf{B}}{\partial t}\right\} d\mathbf{r} + \oint_S \mathbf{S}\, d\mathbf{s}, \tag{2}$$

where $d\mathbf{s}$ is the element of the surface $S$ surrounding $V$. Eq. (2) expresses the energy loss balance; the work done by the particle against its electric field will be distributed between the absorption of electro-magnetic energy in the surrounding volume $V$ and the energy flux over the surface $S$. In an absorbing medium for sufficiently big volume $V$ the second term in the right side of (2) can be neglected, and

$$\frac{d\bar{\Delta}}{dt} = -e\mathbf{v}\mathbf{E}(\mathbf{v}t, t) = \frac{1}{4\pi}\int_V \left\{\mathbf{E}\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}\frac{\partial \mathbf{B}}{\partial t}\right\} d\mathbf{r} = \frac{d\overline{U}}{dt}, \tag{3}$$

where $\bar{\Delta}$ and $\overline{U}$ are the mean energy loss of the particle and absorbed energy in the surrounding volume, respectively.

In the next sections we will derive the energy-angular distributions of the mean number of secondary quasi-particles (like photons in the medium, plasmons or $\delta$-electrons) produced by a relativistic charge moving with arbitrary acceleration in a non-transparent medium, $\epsilon_2 > 0$. Instead of using of the Frank–Tamm–Fermi method [6, 7] based on the calculation of the energy loss as the flux of the Poynting vector over a cylindrical surface surrounding the particle trajectory, we will use the Landau method [4]. The latter is based on the calculation of the energy loss of a relativistic charged particle as the work done by the electric field produced by the particle at its current position. The energy loss is considered to be small compared to the particle energy; therefore the classical electrodynamics calculations can be applied. This approach allows us to derive the energy-angular distribution of primary quasi-particles generated by the incident particle along the particle trajectory.

## 2. General considerations

Following the results of [2,4,8] we calculate the mean radiation energy loss based on the solution of Maxwell's equations for an electro-magnetic field created by a relativistic charged particle moving in a non-magnetic dielectric medium. It is convenient to introduce the vector $\mathbf{A}$ and scalar $\phi$ potentials:

$$\mathbf{H} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla \cdot \phi, \tag{4}$$

in the Coulomb gauge:

$$\nabla \cdot \mathbf{A} = 0. \tag{5}$$

Maxwell's equations for the potentials become:

$$\nabla \cdot (\epsilon \nabla \phi) = -4\pi \varrho(\mathbf{r}, t), \tag{6}$$

$$-\nabla^2 \mathbf{A} = -\frac{1}{c^2}\frac{\partial}{\partial t}\left(\epsilon \frac{\partial \mathbf{A}}{\partial t}\right) - \frac{1}{c}\frac{\partial}{\partial t}(\epsilon \nabla \phi) + \frac{4\pi}{c}\mathbf{j}(\mathbf{r}, t), \tag{7}$$

where $\varrho(\mathbf{r}, t)$ and $\mathbf{j}(\mathbf{r}, t)$ are external charge and electric current densities, respectively. These equations can be solved by introducing the Fourier transforms:

$$F(\mathbf{r}, t) = \iint \frac{d\mathbf{k}\, d\omega}{(2\pi)^4} F(\mathbf{k}, \omega) \exp\bigl[i(\mathbf{kr} - \omega t)\bigr], \tag{8}$$

and

$$F(\mathbf{k}, \omega) = \iint d\mathbf{r}\, dt\, F(\mathbf{r}, t) \exp\bigl[-i(\mathbf{kr} - \omega t)\bigr], \tag{9}$$

where $\hbar\mathbf{k}$ is the momentum transferred from the incident particle to the medium and $\hbar\omega$ is the energy transfer ($\hbar$ is Planck's constant, $\mathbf{k}$ is the wave vector and $\omega$ is the frequency of the emitted photons). The Fourier components of the electric fields according to (4) are:

$$\mathbf{E}_\perp(\mathbf{k}, \omega) = i\frac{4\pi}{c^2}\omega \frac{\mathbf{j}(\mathbf{k}, \omega) - \frac{\mathbf{k}\omega}{k^2}\varrho(\mathbf{k}, \omega)}{k^2 - \epsilon(\mathbf{k}, \omega)\frac{\omega^2}{c^2}}, \qquad \mathbf{E}_\perp(\mathbf{k}, \omega) \cdot \mathbf{k} = 0, \tag{10}$$

and

$$\mathbf{E}_\parallel(\mathbf{k}, \omega) = i\frac{4\pi}{k^2}\mathbf{k}\frac{\varrho(\mathbf{k}, \omega)}{-\epsilon(\mathbf{k}, \omega)}, \qquad \mathbf{E}(\mathbf{k}, \omega) = \mathbf{E}_\perp(\mathbf{k}, \omega) + \mathbf{E}_\parallel(\mathbf{k}, \omega). \tag{11}$$

We will start with the derivation of the full mean radiation energy loss as a function of the transferred energy and emission angle. Consider the mean radiation energy loss $\Delta_\perp$ produced by a relativistic charged particle arbitrarily moving in a non-transparent medium. According to the Landau method [4] it can be expressed as:

$$\begin{aligned}\bar{\Delta}_\perp &= -\int_{-\infty}^{\infty} dt \int_{R_3} d\mathbf{r}\, \mathbf{j}(\mathbf{r}, t) \cdot \mathbf{E}_\perp(\mathbf{r}, t) = U_\perp = \frac{1}{4\pi}\int_{-\infty}^{\infty} dt \int_{R_3} d\mathbf{r}\, \mathbf{E}_\perp(\mathbf{r}, t) \cdot \frac{\partial \mathbf{D}_\perp}{\partial t} \\ &= -\frac{2}{(2\pi)^4}\int_0^{\infty} d\omega \int_{K_3} d\mathbf{k}\, \mathrm{Re}\bigl\{\mathbf{j}^*(\mathbf{k}, \omega) \cdot \mathbf{E}_\perp(\mathbf{k}, \omega)\bigr\},\end{aligned} \tag{12}$$

where $U_\perp$ and $\mathbf{D}_\perp$ are the full absorbed radiation energy and electrical displacement, respectively. Substituting into the last equation the relation (10), we have:

$$\bar{\Delta}_\perp = \frac{1}{2\pi^3}\int_0^\infty \omega\, d\omega \int_{K_3} \frac{d\mathbf{k}}{k^2}\,\mathrm{Im}\left\{\frac{|\mathbf{k}\times\mathbf{j}(\mathbf{k},\omega)|^2}{c^2[k^2-\epsilon(\mathbf{k},\omega)\frac{\omega^2}{c^2}]}\right\}. \tag{13}$$

The corresponding relation for the longitudinal (Bohr) energy loss will be:

$$\bar{\Delta}_\parallel = \frac{1}{2\pi^3}\int_0^\infty \omega\, d\omega \int_{K_3} \frac{d\mathbf{k}}{k^2}\,\mathrm{Im}\left\{\frac{|\varrho(\mathbf{k},\omega)|^2}{-\epsilon(\mathbf{k},\omega)}\right\}. \tag{14}$$

The latter expression describes the energy loss producing longitudinal excitations (plasmons, $\delta$-electrons) of the medium.

The energy spectrum of emitted photons is defined as:

$$\frac{d\bar{\Delta}_\perp}{\hbar\, d\omega} = \hbar\omega\frac{d\overline{N}}{\hbar\, d\omega},$$

where $\overline{N}$ is the mean number of emitted photons.

For a point-like charge $e$ moving along the trajectory $\mathbf{r}(t)$ the Fourier component of the electric current density $\mathbf{j}(\mathbf{k},\omega)$ reads:

$$\mathbf{j}(\mathbf{k},\omega) = e\int_{\mathbf{v}\neq 0} dt\,\mathbf{v}(t)\exp\bigl\{i\bigl[\omega t - \mathbf{k}\mathbf{r}(t)\bigr]\bigr\}, \tag{15}$$

where $\mathbf{v}(t)$ is the time variable velocity of the charge. Then the emitted radiation energy is proportional to the following expression:

$$\bar{\Delta}_\perp \sim \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\tau\,\bigl[k^2\mathbf{v}(t+\tau)\mathbf{v}(t)-\omega^2\bigr]\exp\bigl\{i\omega\tau - i\mathbf{k}\bigl[\mathbf{r}(t+\tau)-\mathbf{r}(t)\bigr]\bigr\},$$

where the time $t$ can be treated as the time along the particle trajectory and we have used the continuity equation, $\mathbf{k}\cdot\mathbf{j}(\mathbf{k},\omega) = \omega\varrho(\mathbf{k},\omega)$. It allows us to introduce the intensity of radiation $\overline{W}_\perp$:

$$\begin{aligned}\overline{W}_\perp(t) &= \frac{d\bar{\Delta}_\perp}{dt}\\ &= \frac{1}{2\pi^3}\int_0^\infty \omega\, d\omega\,\mathrm{Im}\left\{\int_{K_3}\frac{d\mathbf{k}}{k^2c^2[k^2-\epsilon(\mathbf{k},\omega)\frac{\omega^2}{c^2}]}\int_{-\infty}^{\infty} d\tau\,\bigl[k^2\mathbf{v}(t+\tau)\mathbf{v}(t)-\omega^2\bigr]\right.\\ &\qquad \left.\times\exp\bigl\{i\omega\tau - i\mathbf{k}\bigl[\mathbf{r}(t+\tau)-\mathbf{r}(t)\bigr]\bigr\}\right\}.\end{aligned} \tag{16}$$

Since for the arbitrary vector $\mathbf{A}$:

$$\int_{4\pi} d\Omega\,\exp\{i\mathbf{n}\cdot\mathbf{A}\} = 4\pi\frac{\sin|\mathbf{A}|}{|\mathbf{A}|}, \qquad \mathbf{k} = k\mathbf{n},$$

the spectral intensity can be written as:

$$\frac{d\overline{W}_\perp(t)}{\hbar\, d\omega} = \frac{2\alpha}{\pi^2}\frac{\omega}{c}\,\mathrm{Im}\left\{\int_0^\infty \frac{dk}{k[k^2-\epsilon(\mathbf{k},\omega)\frac{\omega^2}{c^2}]}\int_{-\infty}^{\infty} d\tau\,\frac{k^2\mathbf{v}(t+\tau)\mathbf{v}(t)-\omega^2}{|\mathbf{r}(t+\tau)-\mathbf{r}(t)|} \times \exp\{i\omega\tau\}\sin\bigl[k\bigl|\mathbf{r}(t+\tau)-\mathbf{r}(t)\bigr|\bigr]\right\}, \tag{17}$$

where $\alpha$ is the fine structure constant.

In the next sections we consider some practical examples illustrating the applications of (13)–(17).

## 3. Cherenkov radiation

In the case of Cherenkov radiation charge $e$ experiences uniform motion with constant velocity $\mathbf{v}$ along an infinite trajectory. It allows us to calculate the integral with respect to $\tau$ in (16):

$$\int_{-\infty}^{\infty} d\tau\,\bigl[k^2\mathbf{v}(t+\tau)\mathbf{v}(t)-\omega^2\bigr]\exp\bigl\{i\omega\tau - i\mathbf{k}\bigl[\mathbf{r}(t+\tau)-\mathbf{r}(t)\bigr]\bigr\} = 2\pi k^2 v^2 \sin^2\theta\,\delta(\mathbf{k}\cdot\mathbf{v}-\omega),$$

where $\delta$ is the Dirac delta function and $\theta$ is the angle between $\mathbf{v}$ and $\mathbf{k}$. Then the radiation intensity reads:

$$\overline{W}_\perp = \frac{2e^2}{\pi}\beta^2\int_0^\infty \omega\, d\omega\int_{-1}^{1}\sin^2\theta\, d\cos\theta\int_0^\infty dk\,\mathrm{Im}\left\{\frac{1}{1-\epsilon(\mathbf{k},\omega)\frac{\omega^2}{c^2k^2}}\right\}\delta(\mathbf{k}\cdot\mathbf{v}-\omega), \tag{18}$$

where $\beta = v/c$. The delta function allows us to perform the integration with respect to $k$. Introducing the element of the particle trajectory $dx = v\,dt$, we have:

$$\frac{d\bar{\Delta}_\perp}{dx} = \frac{2e^2}{\pi c^2}\int_0^\infty \omega\, d\omega\int_0^1 \frac{\sin^2\theta}{\cos\theta}\, d\cos\theta\,\mathrm{Im}\left\{\frac{1}{1-\epsilon\beta^2\cos^2\theta}\right\}. \tag{19}$$

Expanding the imaginary part we get for the energy-angular spectrum of emitted photons [9]:

$$\frac{d^3\overline{N}_\perp}{\hbar\, d\omega\, dx\, d\cos^2\theta} = \frac{\alpha}{\hbar c}\sin^2\theta\,\frac{\Gamma}{\pi[(\cos^2\theta-\cos^2\theta_o)^2+\Gamma^2]}, \tag{20}$$

where,

$$\Gamma = \frac{\epsilon_2}{\beta^2|\epsilon|^2}, \tag{21}$$

and

$$\cos^2\theta_o = \frac{\epsilon_1}{\beta^2|\epsilon|^2}. \tag{22}$$

The distribution (20) clearly shows that in an absorbing medium the angular distribution of Cherenkov photons experiences additional broadening even at fixed photon energy. The distribution has a sharp peak at $\cos^2\theta = \cos^2\theta_o$. Therefore the most probable emission angle $\theta_o$ of Cherenkov photons in a non-transparent medium is defined by:

$$\cos\theta_o = \frac{\sqrt{\epsilon_1}}{\beta|\epsilon|} \simeq \frac{1-\frac{\nu^2}{2}}{\beta\sqrt{\epsilon_1}}, \qquad \nu = \frac{\Gamma}{\cos^2\theta_o} = \frac{\epsilon_2}{\epsilon_1}. \tag{23}$$

The full width at half maximum, FWHM, of the peak is equal to:

$$\mathrm{FWHM} = 2\Gamma = \frac{2\epsilon_2}{\beta^2|\epsilon|^2}. \tag{24}$$

The integration of (20) with respect to $\cos^2\theta$ results in the energy spectrum of the radiation (we consider the dielectric permittivity as a function of the photon frequency only and in the visible-ultraviolet range of interest to be independent of $\cos\theta$ (no spatial dispersion), $\epsilon = \epsilon(\omega)$):

$$\frac{d^2\overline{N}_\perp}{\hbar\, d\omega\, dx} = \frac{\alpha}{\pi\hbar c}\,\mathrm{Im}\left\{\left[1 - \frac{1}{\beta^2\epsilon}\right]\ln\left(\frac{1}{1-\beta^2\epsilon}\right)\right\}. \tag{25}$$

In the limit of a transparent medium, when $\Gamma \to 0$, we note that

$$\frac{\Gamma}{\pi[(\cos^2\theta - \cos^2\theta_o)^2 + \Gamma^2]} \to \delta\left(\cos^2\theta - \frac{1}{\beta^2 n^2}\right),$$

and get the energy-angular distribution of the mean number of Cherenkov photons:

$$\frac{d^3\overline{N}_\perp}{\hbar\, d\omega\, dx\, d\cos^2\theta} = \alpha\frac{1}{\hbar c}\left(1 - \frac{1}{\beta^2 n^2}\right)\delta\left(\cos^2\theta - \frac{1}{\beta^2 n^2}\right). \tag{26}$$

This expression fixes explicitly the emission angle to be $\theta_c = \arccos(1/\beta n)$. If $\beta n > 1$, integration of (26) with respect to $\cos^2\theta$ results in the well-known formula of the Frank–Tamm theory [6]:

$$\frac{d^2\overline{N}_\perp}{\hbar\, d\omega\, dx} = \frac{\alpha}{\hbar c}\left(1 - \frac{1}{\beta^2 n^2}\right), \quad \beta n > 1. \tag{27}$$

In the optical-ultraviolet range the value of $\nu$ can be estimated from:

$$\nu \sim \frac{c}{n\omega l} \sim \frac{\lambda}{l}, \tag{28}$$

where $\lambda$ is the photon wavelength, and $l$ is the photon absorption length. Since for semi-transparent media $\hbar\omega l \gtrsim 2$ eV cm and $\hbar c \sim 2\times 10^{-5}$ eV cm,

$$\nu \lesssim 10^{-5},$$

and the broadening is in practice very difficult to observe in semi-transparent solids and liquids. Fig. 1 illustrates this showing the angle distributions of Cherenkov photons produced in $C_6F_{14}$ [9] by relativistic particle $v \sim c$ for two values of the photon energy: $\hbar\omega = 6.8$ eV (open circles) and $\hbar\omega = 7.7$ eV (closed circles and upper $x$-axis). The curves are normalised to the equal most probable values.

For gases the situation can be more promising. Here the Cherenkov angle is always very small, $\theta \ll 1$ or $\cos^2\theta_o \sim 1$. Then the relative broadening due to the gas absorption $\delta\theta/\theta$ can be estimated from:

$$\frac{\delta\theta}{\theta} \sim \frac{\nu}{\theta^2} \sim \frac{c}{2\eta\omega l} \sim \frac{\lambda}{\eta l}, \tag{29}$$

where $\eta = n - 1$ is the refractivity. In most gases of practical use $\eta \sim 10^{-4}$ and therefore the aberration of Cherenkov radiation due to absorption can be observed in gas ring imaging Cherenkov (RICH) detectors [10]. Radio frequency Cherenkov photons (meter range) can be also considered as promising for the observation of the broadening due to absorption since $\nu$ is proportional to the ratio of the wavelength to the absorption length of the photons.

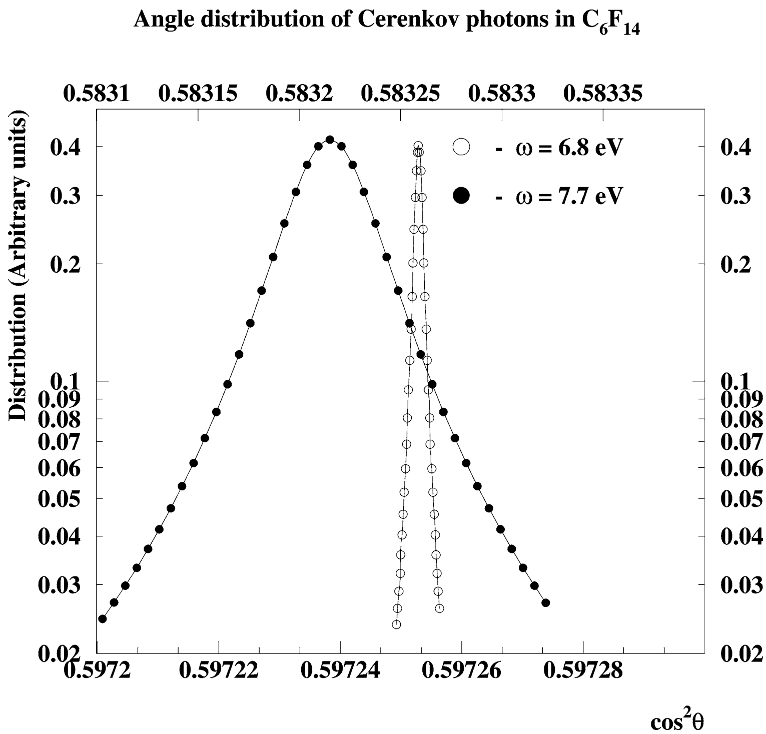


Fig. 1. The angular distributions of Cherenkov photons produced in $C_6F_{14}$ by a relativistic particle $v \sim c$ for two values of the photon energy: $\hbar\omega = 6.8$ eV (open circles) and $\hbar\omega = 7.7$ eV (closed circles and upper $x$-axis). The curves are normalised to equal most probable values.

## 4. Prompt bremsstrahlung

It is interesting to apply the general relation (13) for a description of the semi-infinite motion of the charged particle $e$ with constant velocity $\mathbf{v}$. This case corresponds, for example, to the beta-decay of a nucleus, when the electron starts to move in the surrounding medium. The Fourier component of the electric current density reads:

$$\mathbf{j}(\mathbf{k},\omega) = e\mathbf{v}\int_0^\infty dt\,\exp\bigl\{i[\omega t - \mathbf{k}\mathbf{v}]t\bigr\} = e\mathbf{v}\left[\pi\delta(\mathbf{k}\cdot\mathbf{v}-\omega) - \mathrm{V_P}\frac{i}{\mathbf{k}\cdot\mathbf{v}-\omega}\right], \tag{30}$$

where $\mathrm{V_P}$ means the principal value. Then:

$$\bigl|\mathbf{k}\times\mathbf{j}(\mathbf{k},\omega)\bigr|^2 = e^2k^2v^2\sin^2\theta\left[\pi T\delta(\mathbf{k}\cdot\mathbf{v}-\omega) + \frac{1}{(\mathbf{k}\cdot\mathbf{v}-\omega)^2}\right],$$

where $T$ is proportional to the full time of the motion. The first term in the latter relation corresponds to Cherenkov radiation produced from the semi-infinite trajectory, while the second one is responsible for the radiation from initial prompt acceleration (bremsstrahlung). Let us consider the full mean radiation energy loss due to prompt bremsstrahlung. The corresponding part of relation (13) reads:

$$\bar{\Delta}_\perp = \frac{e^2}{\pi^2c^2}\int_0^\infty \omega\,d\omega\int_0^1\frac{\sin^2\theta}{\cos^2\theta}\,d\cos\theta\int_0^\infty k^2\,dk\,\mathrm{Im}\left\{\frac{1}{(k^2-a^2)(k-b)^2} + \frac{1}{(k^2-a^2)(k+b)^2}\right\}, \tag{31}$$

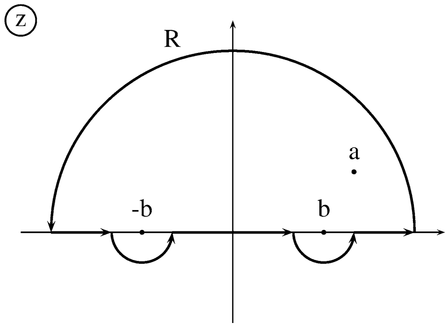


Fig. 2. The contour of integration for the prompt bremsstrahlung.

where the following notation was introduced:

$$a^2 = \epsilon \frac{\omega^2}{c^2}, \qquad b = \frac{\omega}{v \cos\theta}.$$

The relation inside the imaginary part is an even function of $k$. Therefore it is convenient to perform the integration with respect to $k$ in the complex plane $z$ over the contour shown in Fig. 2:

$$\bar{\Delta}_\perp = \frac{e^2}{\pi^2 c^2} \int_0^\infty \omega\, d\omega \int_0^1 \frac{\sin^2\theta}{\cos^2\theta}\, d\cos\theta\, \mathrm{Im}\left\{ \oint_C f(z)\, dz \right\}, \tag{32}$$

where $f(z)$ is defined by:

$$f(z) = \frac{z^2}{2(z^2 - a^2)} \left[ \frac{1}{(z-b)^2} + \frac{1}{(z+b)^2} \right],$$

and:

$$\lim_{R\to\infty} \int_R f(z)\, dz = 0.$$

The integral over the contour $C$ is defined by the sum of residues in the singularities of $f(z)$, i.e., the points $a$ and $\pm b$. Since:

$$\mathrm{res}(-b) + \mathrm{res}(b) = 0,$$

the contour integral reads:

$$\mathrm{Im}\left\{ \oint_C f(z)\, dz \right\} = 2\pi\, \mathrm{Re}\{\mathrm{res}(a)\} = \frac{\pi}{2} \frac{v^2 \cos^2\theta}{\omega c} \mathrm{Re}\left\{ \frac{\sqrt{\epsilon}}{(1 - \beta\sqrt{\epsilon}\cos\theta)^2} + \frac{\sqrt{\epsilon}}{(1 + \beta\sqrt{\epsilon}\cos\theta)^2} \right\}. \tag{33}$$

Substituting the latter relation into Eq. (32) results in the energy-angular distribution of the prompt bremsstrahlung energy loss:

$$\frac{d^2 \bar{\Delta}_\perp}{\hbar\, d\omega\, d\cos\theta} = \frac{\alpha}{2\pi} \beta^2 \sin^2\theta\, \mathrm{Re}\left\{ \frac{\sqrt{\epsilon}}{(1 - \beta\sqrt{\epsilon}\cos\theta)^2} \right\}. \tag{34}$$

The latter relation reduces in the case of vacuum to the well-known expression (see, for example, [11]):

$$\frac{d\bar{\Delta}_\perp}{\hbar\, d\omega} = \frac{\alpha}{2\pi} \beta^2 \int_0^\pi \frac{\sin^3\theta\, d\theta}{(1 - \beta\cos\theta)^2}. \tag{35}$$

For the description of prompt bremsstrahlung in the case that the charge $e$ instantly changes its mode of uniform motion from velocity $\mathbf{v}_1$ to $\mathbf{v}_2$, one has to investigate the generalised version of (31):

$$\frac{d\bar{\Delta}_\perp}{\hbar\, d\omega} = \frac{\alpha}{2\pi^3}\frac{\omega}{c}\int\limits_{K_3} d\mathbf{k}\,\mathrm{Im}\left\{\frac{\left[\frac{\mathbf{n}\times\mathbf{v}_1}{\mathbf{k}\cdot\mathbf{v}_1-\omega} - \frac{\mathbf{n}\times\mathbf{v}_2}{\mathbf{k}\cdot\mathbf{v}_2-\omega}\right]^2}{k^2-\epsilon(\mathbf{k},\omega)\frac{\omega^2}{c^2}}\right\}, \tag{36}$$

where $\mathbf{n}$ is the unit vector in the direction of $\mathbf{k}$.

## 5. Uniform motion over finite distance. The Tamm problem

Let us consider now the uniform motion of the charge $e$ with velocity $\mathbf{v}$ along a finite distance (Fig. 3). This problem was considered for transparent medium by Tamm [12] and is referred to as *the Tamm problem*. We can again calculate easily the integral with respect to $\tau$ in (16):

$$\int\limits_{-T-t}^{T-t} d\tau\left[k^2\mathbf{v}(t+\tau)\mathbf{v}(t)-\omega^2\right]\exp\left\{i\omega\tau - i\mathbf{k}\left[\mathbf{r}(t+\tau)-\mathbf{r}(t)\right]\right\}$$
$$= 2k^2v^2\sin^2\theta\exp\left[i(\mathbf{k}\cdot\mathbf{v}-\omega)t\right]\frac{\sin[(\mathbf{k}\cdot\mathbf{v}-\omega)T]}{\mathbf{k}\cdot\mathbf{v}-\omega},$$

where $2T$ is the full time duration of the motion. The spectral-angular distribution of the radiation intensity reads:

$$\frac{d^2\overline{W}_\perp(t)}{\hbar\, d\omega\, d\Omega} = \frac{\alpha}{\pi^3}\frac{\omega}{c}v^2\sin^2\theta\,\mathrm{Im}\left\{\int\limits_0^\infty k^2\,dk\frac{\exp[i(\mathbf{k}\cdot\mathbf{v}-\omega)t]\sin[(\mathbf{k}\cdot\mathbf{v}-\omega)T]}{[k^2-\epsilon(\mathbf{k},\omega)\frac{\omega^2}{c^2}](\mathbf{k}\cdot\mathbf{v}-\omega)}\right\}. \tag{37}$$

For $T\to\infty$, since:

$$\frac{T}{\pi}\frac{\sin(Tx)}{Tx}\to\delta(x),\qquad T\to\infty,$$

the spectral-angle distribution of the radiation intensity corresponds to the case of Cherenkov radiation (18):

$$\frac{d^2\overline{W}_\perp}{\hbar\, d\omega\, d\Omega} = \frac{\alpha}{\pi^2}\frac{\omega}{c}v^2\sin^2\theta\,\mathrm{Im}\left\{\int\limits_0^\infty k^2\,dk\frac{\delta(\mathbf{k}\cdot\mathbf{v}-\omega)}{k^2-\epsilon(\mathbf{k},\omega)\frac{\omega^2}{c^2}}\right\}.$$

Let us consider the full mean radiation energy loss for uniform motion along a finite distance $2vT$, which reads:

$$\bar{\Delta}_\perp = \frac{4e^2}{\pi^2c^2}\int\limits_0^\infty\omega\,d\omega\int\limits_0^1\frac{\sin^2\theta}{\cos^2\theta}\,d\cos\theta\int\limits_0^\infty k^2\,dk\,\mathrm{Im}\left\{\frac{\sin^2[d(k-b)]}{(k^2-a^2)(k-b)^2}+\frac{\sin^2[d(k+b)]}{(k^2-a^2)(k+b)^2}\right\}, \tag{38}$$

where we introduced the following notation:

$$a^2 = \epsilon\frac{\omega^2}{c^2},\qquad b=\frac{\omega}{v\cos\theta},\qquad d = vT\cos\theta.$$

The relation inside the imaginary part is an even function of $k$. Therefore it is convenient to perform the integration with respect to $k$ in the complex plane $z$ over the contour shown in Fig. 4:

$$\bar{\Delta}_\perp = \frac{4e^2}{\pi^2c^2}\int\limits_0^\infty\omega\,d\omega\int\limits_0^1\frac{\sin^2\theta}{\cos^2\theta}\,d\cos\theta\,\mathrm{Im}\left\{\oint\limits_C g(z)\,dz\right\}, \tag{39}$$

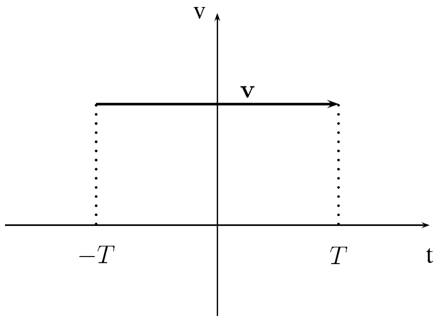


Fig. 3. Uniform motion along finite distance (the Tamm problem).

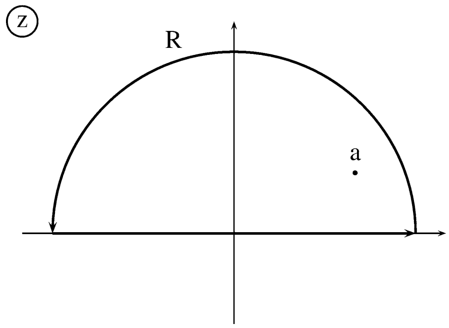


Fig. 4. The contour of integration for the Tamm problem.

where $g(z)$ is defined by:

$$g(z) = \frac{z^2}{2(z^2 - a^2)} \left\{ \frac{\sin^2[d(z-b)]}{(z-b)^2} + \frac{\sin^2[d(z+b)]}{(z+b)^2} \right\},$$

and:

$$\lim_{R\to\infty} \int\limits_R g(z)\, dz = 0.$$

The integral over the contour $C$ is defined by the residue in the singularity of $g(z)$, i.e., at the point $a$. The contour integral reads:

$$\begin{aligned} \mathrm{Im}\left\{ \oint\limits_C g(z)\, dz \right\} &= 2\pi\, \mathrm{Re}\{\mathrm{res}(a)\} \\ &= \frac{\pi}{2} \frac{v^2 \cos^2\theta}{\omega c} \mathrm{Re}\left\{ \frac{\sqrt{\epsilon}\sin^2[\omega T(1-\beta\sqrt{\epsilon}\cos\theta)]}{(1-\beta\sqrt{\epsilon}\cos\theta)^2} + \frac{\sqrt{\epsilon}\sin^2[\omega T(1+\beta\sqrt{\epsilon}\cos\theta)]}{(1+\beta\sqrt{\epsilon}\cos\theta)^2} \right\}. \end{aligned} \tag{40}$$

Substituting the latter relation into Eq. (39) results in the energy-angular distribution of the radiation energy loss for the Tamm problem in absorbing medium:

$$\frac{d^2\bar{\Delta}_\perp}{\hbar\, d\omega\, d\cos\theta} = \frac{2\alpha}{\pi}\beta^2 \sin^2\theta\, \mathrm{Re}\left\{ \frac{\sqrt{\epsilon}\sin^2[\omega T(1-\beta\sqrt{\epsilon}\cos\theta)]}{(1-\beta\sqrt{\epsilon}\cos\theta)^2} \right\}. \tag{41}$$

The similar calculations for the radiation intensity result in:

$$\frac{d^2\overline{W}_\perp(t)}{\hbar\, d\omega\, d\cos\theta} = \frac{\alpha}{\pi}\beta^2\omega\sin^2\theta\,\mathrm{Re}\left\{\sqrt{\epsilon}\exp\bigl[-i\omega t(1-\beta\sqrt{\epsilon}\cos\theta)\bigr]\frac{\sin[\omega T(1-\beta\sqrt{\epsilon}\cos\theta)]}{1-\beta\sqrt{\epsilon}\cos\theta}\right\}. \tag{42}$$

This expression is the full solution of the Tamm problem in an absorbing medium. Note that the integration of (42) with respect to $t$ in the limits $(-T, T)$ results in (41). In a transparent medium expression (41) reduces to the results of [12].

## 6. Undulator-like radiation. Doppler effect in absorbing medium

Let a charge $e$ move according the following law:

$$\mathbf{r}_o(t) = \mathbf{v}t + \mathbf{a}\sin(\omega_o t), \qquad \mathbf{v}_o(t) = \mathbf{v} + \omega_o\mathbf{a}\cos(\omega_o t), \tag{43}$$

where $\mathbf{v}$ is the velocity of uniform motion, $\mathbf{a}$ and $\omega_o$ are the amplitude and the frequency of the periodic part of the motion, respectively. We consider $\mathbf{a}$ to be small (dipole approximation, $a = |\mathbf{a}| \ll v/\omega_o$), so that in the expression for the Fourier component of the electric current density we can expand the exponential:

$$\begin{aligned}\mathbf{j}(\mathbf{k},\omega) &= e\int_{-\infty}^{\infty} dt\,\mathbf{v}_o(t)\exp\bigl\{i\bigl[\omega t - \mathbf{k}\mathbf{r}_o(t)\bigr]\bigr\} \\ &\simeq e\int_{-\infty}^{\infty} dt\,\mathbf{v}_o(t)\bigl[1 - i\mathbf{k}\cdot\mathbf{a}\sin(\omega_o)t\bigr]\exp\bigl\{i[\omega t - \mathbf{k}\mathbf{v}]t\bigr\} \\ &= 2\pi e\bigl\{\mathbf{v}\delta(\mathbf{k}\cdot\mathbf{v}-\omega) + \mathbf{A}_+\delta(\mathbf{k}\cdot\mathbf{v}-\omega-\omega_o) + \mathbf{A}_-\delta(\mathbf{k}\cdot\mathbf{v}-\omega+\omega_o)\bigr\},\end{aligned} \tag{44}$$

where the amplitudes $\mathbf{A}_\pm$:

$$\mathbf{A}_\pm = \frac{1}{2}\bigl[\omega_o\mathbf{a} \mp \mathbf{v}(\mathbf{k}\cdot\mathbf{a})\bigr].$$

For $\omega \gg \omega_o$ and $v \sim c$:

$$\mathbf{A}_\pm \sim \mp\frac{\mathbf{v}(\mathbf{k}\cdot\mathbf{a})}{2}.$$

Then we have:

$$\bigl|\mathbf{k}\times\mathbf{j}(\mathbf{k},\omega)\bigr|^2 = 2\pi e^2 v^2\sin^2\theta T k^2\left\{\delta(\mathbf{k}\cdot\mathbf{v}-\omega) + \frac{(\mathbf{k}\cdot\mathbf{a})^2}{4}\bigl[\delta(\mathbf{k}\cdot\mathbf{v}-\omega-\omega_o) + \delta(\mathbf{k}\cdot\mathbf{v}-\omega+\omega_o)\bigr]\right\}, \tag{45}$$

where $T$ is proportional to the full time of motion.

The case $\mathbf{a} \parallel \mathbf{v}$ considered below corresponds, for example, to the movement through a stack of charged capacities. Repeating the consideration of Section 3, we get the energy-angular spectrum for the mean number of photons emitted per unit trajectory length $dx = v\,dt$:

$$\begin{aligned}&\frac{d^3\overline{N}_\perp}{\hbar\, d\omega\, dx\, d\cos^2\theta} \\ &\quad= \frac{\alpha}{\pi\hbar c}\bigl(1-\cos^2\theta\bigr)\left\{\frac{\Gamma}{(\cos^2\theta-\cos^2\theta_o)^2+\Gamma^2} + \frac{(\frac{a\omega}{2vd_+})^2\Gamma_+}{(\cos^2\theta-\cos^2\theta_+)^2+\Gamma_+^2}\right. \\ &\qquad\left. + \frac{(\frac{a\omega}{2vd_-})^2\Gamma_-}{(\cos^2\theta-\cos^2\theta_-)^2+\Gamma_-^2}\right\},\end{aligned} \tag{46}$$

where the following notation was introduced:

$$d_\pm^2 = \left(\frac{\omega}{\omega \pm \omega_o}\right)^2, \qquad \Gamma = \frac{\epsilon_2}{\beta^2|\epsilon|^2}, \qquad \Gamma_\pm = \frac{\epsilon_2}{\beta^2|\epsilon|^2 d_\pm^2}, \tag{47}$$

and

$$\cos^2\theta_o = \frac{\epsilon_1}{\beta^2|\epsilon|^2}, \qquad \cos^2\theta_\pm = \frac{\epsilon_1}{\beta^2|\epsilon|^2 d_\pm^2}. \tag{48}$$

This expression defines the Doppler effect (second and third terms) for undulator radiation in the dipole approximation in an absorbing medium. It is seen that because of the absence of the singularity at the Cherenkov cone, the Doppler effect does not subdivide into regions of the normal and anomalous effects. In the limit of a transparent medium $\epsilon_2 \to 0$ Eq. (46), however, reduces to:

$$\begin{aligned}\frac{d^3\overline{N}_\perp}{\hbar\, d\omega\, dx\, d\cos^2\theta} &= \frac{\alpha}{\hbar c}\left(1-\cos^2\theta\right)\left\{\delta\left(\cos^2\theta - \cos^2\theta_c\right) + \left(\frac{a\omega}{2vd_+}\right)^2 \delta\left(\cos^2\theta - \cos^2\theta_+\right)\right. \\ &\quad \left. + \left(\frac{a\omega}{2vd_-}\right)^2 \delta\left(\cos^2\theta - \cos^2\theta_-\right)\right\},\end{aligned} \tag{49}$$

where the first term is responsible for Cherenkov radiation, while the second and third terms describe the Doppler effect in a transparent medium:

$$\cos\theta_c = \frac{1}{\beta n}, \qquad \cos\theta_\pm = \frac{\omega \pm \omega_o}{\beta n\omega} = \left(1 \pm \frac{\omega_o}{\omega}\right)\cos\theta_c.$$

The latter equation describes the energy-angular distribution of photons emitted by relativisitic charge moving according to (43) in a transparent medium. The singularity of the angle distribution at $\cos\theta_c = 1/\beta n$ separates the regions into anomalous:

$$\omega = \frac{\omega_o}{\beta n\cos\theta - 1}, \quad \text{for } \cos\theta > \cos\theta_c,$$

and normal:

$$\omega = \frac{\omega_o}{1 - \beta n\cos\theta}, \quad \text{for } \cos\theta < \cos\theta_c,$$

Doppler effects (see, Fig. 5, reviews [5,13,14], and references therein). It should be emphasised that this simple picture of the Doppler effect is valid under the dipole radiation condition (44) only. In the case of bigger $a$ there will appear additional angular maxima corresponding to higher harmonics at:

$$\cos\theta_{\pm\nu} = \frac{\omega \pm \nu\omega_o}{\beta n\omega} = \left(1 \pm \frac{\nu\omega_o}{\omega}\right)\cos\theta_c,$$

where $\nu > 1$ is an arbitrary integer number. Qualitatively, it means that the Cherenkov cone will be surrounded by a halo of Doppler-line satellites.

## 7. High-energy, low-angle approximation

It is interesting to consider the X-ray range of the radiation produced by relativistic charged particles in an absorbing medium. For frequencies more than $K$-shell excitation potential we can use the standard high frequency

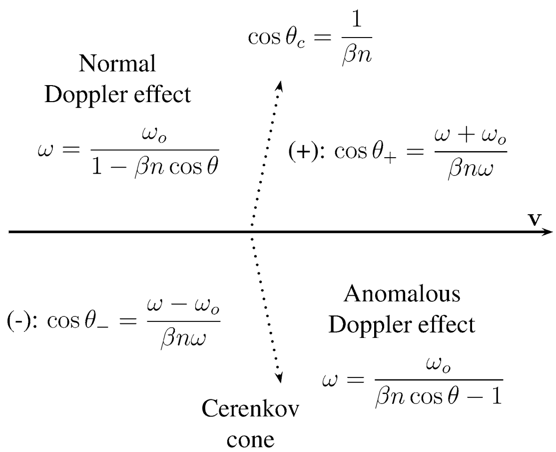


Fig. 5. The regions of anomalous and normal Doppler effects in a transparent medium.

approximation for the dielectric permittivity:

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2} + i\frac{c}{\omega l}, \tag{50}$$

where $\omega_p$ and $l$ are the plasma frequency and the photon absorption length in the medium, respectively. Then the mean number of X-ray Cherenkov photons (see also [15]) emitted from unit distance will be (the small angle approximation of (20)):

$$\frac{d^3\overline{N}_{\mathrm{XCR}}}{\hbar\, d\omega\, dx\, d\theta^2} = \frac{\alpha}{\pi\hbar c}\frac{\omega}{c}\theta^2\,\mathrm{Im}\{Z\}, \tag{51}$$

where we introduce the *complex formation zone*, $Z$, of X-ray Cherenkov radiation in the medium:

$$Z = \frac{L}{1 - i\frac{L}{l}}. \tag{52}$$

In the case of a transparent medium $l = \infty$, the complex formation zone is reduced to the *coherence length* $L$ of XCR:

$$L = \frac{c}{\omega}\left[\gamma^{-2} + \frac{\omega_p^2}{\omega^2} + \theta^2\right]^{-1}, \quad \gamma^{-2} = 1 - \beta^2. \tag{53}$$

One can see that X-ray Cherenkov radiation is strongly suppressed in media with small X-ray absorption. For the case of undulator-like radiation the X-ray radiation will be defined by the normal Doppler effect. Neglecting medium absorption in the X-ray range, one has:

$$\frac{d^2\overline{N}}{\hbar\, d\omega\, dx} = \frac{\alpha}{\hbar c}\frac{a^2\omega^2}{4v^2}\left(1 - \frac{\omega_o}{\omega}\right)^2\left[2\frac{\omega_o}{\omega} - \gamma^{-2} - \frac{\omega_p^2}{\omega^2}\right]. \tag{54}$$

X-ray radiation can be observed for frequencies satisfying:

$$\gamma\omega_p \lesssim \omega \lesssim \gamma^2\omega_o, \quad \omega_p \ll \gamma\omega_o.$$

For the case of prompt bremsstrahlung the number of X-ray photons produced by ultra-relativistic $\beta \sim 1$ charge reads:

$$\frac{d^2\overline{N}_{\mathrm{XBR}}}{\hbar\, d\omega\, d\theta^2} = \frac{\alpha}{\pi\hbar c}\frac{\omega}{c}\theta^2\,\mathrm{Re}\{Z^2\}. \tag{55}$$

Note that this relation is quite similar to the the mean number of X-ray transition radiation photons, emitted when the charge $e$ crosses the interface between two media with different dielectric properties:

$$\frac{d^2\overline{N}_{\mathrm{XTR}}}{\hbar\, d\omega\, d\theta^2} = \frac{\alpha}{\pi\hbar c}\frac{\omega}{c}\theta^2\,\mathrm{Re}\bigl\{(Z_1 - Z_2)^2\bigr\}, \tag{56}$$

where $Z_1$ and $Z_2$ are the complex formation zones in the first and the second medium, respectively [16]. Similar expressions can be derived for other modes of relativistic charge motion in absorbing medium. It means, therefore, that the complex formation zone defines the main properties of X-ray (gamma) radiation produced by relativistic charged particle in an absorbing medium.

## 8. Summary

Examples of the radiation energy loss produced by a relativistic charged particle moving with arbitrary acceleration in an absorbing medium allow us to make the following conclusions:

(1) The approach based on the calculation of radiation energy loss looks simple and powerful compared to the estimation of the Poynting vector. The calculation of the radiation energy loss allows us to apply the powerful apparatus of analysis on the complex plane.
(2) In the high-energy low-angle approximation the radiation energy loss can be expressed in terms of the complex formation zone. The latter covers both coherence and absorption effects in the X-ray and gamma range.
(3) The radiation energy loss approach allows us to simulate the emitted photons along the trajectory of the initial charged particle. It is convenient for the simulation of the radiation produced by a relativistic charged particle moving in detectors with complex geometry.

The approach described in this Letter is under implementation in the framework of the GEANT4 tool kit for simulation in high energy physics, astrophysics and medical imaging [3].

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